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What's Wrong with College Algebra?

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What's Wrong with College Algebra?

Each year, more than 1,000,000 students take college algebra and related courses [8]. At many schools, these are the cash cows of the mathematics department, if not the entire institution—there are huge numbers of students (it is usually the largest credit-bearing course in mathematics) and typically one of the very cheapest courses to offer, since it is staffed primarily by part-time faculty in the two- and four-year colleges and by TAs in the universities.

And yet, we in the mathematics community are hearing an increasing cry from all sides that there are major problems with these courses and that they need to be changed. So, what's wrong with college algebra?

Major changes have taken place in the mathematical education of students over the last 10 to 15 years. These changes have come about for a variety of reasons, including

(1) The changing demographics of the students taking college-level mathematics

(2) The growth of technology and what it can provide for the teaching and learning of mathematics,

(3) The changing mathematical needs among the people who use mathematics.

In addition, significant changes are taking place in the schools and the students coming out of those programs, particularly those in New York State, have very different experiences in mathematics and, as a result, different expectations of what and how mathematics should be taught.

All of these factors have major implications for what we teach, and how we teach it, particular at the college algebra and precalculus level, though it is also true in all other mathematics offerings. And it is these implications that have led to so many calls for a change in the focus in these courses so that they no longer focus so heavily on the development of algebraic skills.

For instance, the Mathematical Association of America (MAA) recently published its Curriculum Guide 2004 [11] in which it recommends that

• *All students, those for whom the (introductory mathematics) course is terminal and those for* *whom it serves as a springboard, need to learn to think effectively, quantitatively and logically.*

• *Students must learn with understanding, focusing on relatively few concepts but treating them in depth. Treating ideas in depth includes presenting each concept from multiple points of view and in progressively more sophisticated contexts.*

• *A study of these (disciplinary) reports and the textbooks and curricula of courses in other disciplines shows that the algorithmic skills that are the focus of computational college algebra courses are much less important than understanding the underlying concepts.*

• *Students who are preparing to study calculus need to develop conceptual understanding as well as computational skills*

If implemented at the college level, these recommendations would establish a smooth transition between school and college mathematics.

Similarly, .in talking about the courses below calculus in its *Crossroads* Standards [2], AMATYC (the American Mathematical Association of Two Year Colleges) says

• *Students will use problem-solving strategies that … should include posting questions, organizing information; drawing diagrams; analyzing situations through trial and error, graphing, and modeling; and drawing conclusions by translating, illustrating and verifying results. The student should be able to communicate and interpret their results.*

• *In general, emphasis on the meaning and use of mathematical ideas must increase, and attention to rote manipulation must decrease.*

• *Faculty should include fewer topics but cover them in greater depth, with greater understanding, and with more flexibility. Such an approach will enable students to adapt to new situations.*

• *Areas that should receive increased attention include the conceptual understanding of mathematical ideas.*

These AMATYC recommendations and exemplary efforts as implementing them are clearly enunciated in the AMATYC volume *Programs Reflecting the Standards* [9].

These recommendations from both MAA and AMATYC are clearly very much in the same spirit as the recommendations in NCTM's (the National Council of Teachers of Mathematics) *Principles and Standards for School Mathematics* [12], which are having a significant impact on mathematics education in the schools from K through 12th grade across the nation. If implemented at the college level, these recommendations would establish a smooth transition between school and college mathematics.

Over the last few years, literally scores of mathematicians have given talks at national AMATYC and MAA meetings during which they have described the problems they have with traditional college algebra and related courses and have discussed all kinds of innovative alternatives they have tried that have proven to be highly successful. AMATYC and MAA are involved in a collaborative initiative to refocus these courses. Details on the status and goals of this initiative are described in [7].

As one aspect of this effort, the current chair of CRAFTY, the MAA's Committee on Curriculum Renewal Across the First Two Years, issued a call to some 1800 MAA liaisons, asking them if their department would care to be involved in a proposed grant project to refocus their college algebra course, compare student performance in the alternative sections to those in traditional sections, and track the students to see how they performed in subsequent courses. Within six days, over 210 departments had responded that they wanted to be part of this project. (Unfortunately, only 11 could be accommodated within the available funding limitations of the grant program.) Clearly, large numbers of schools have come to the realization that there are major problems with traditional college algebra offerings and that there is a need to make dramatic changes in those courses.

In the following sections, we will investigate the rationales behind these calls for change in more detail.

Changes in the student population

We start by putting some things into an historical perspective. Over the last 60 years, (since the end of World War), the population of the United States has roughly doubled. In the same time frame, college enrollments have increased roughly ten-fold. The students who came to college in that era represented a very small portion of the total U.S. population. From a traditional mathematical perspective, they were an elite group who had mastered a high level of proficiency in traditional high school mathematics, particularly algebraic manipulation. They entered college reasonably well prepared for the standard freshman course in calculus, which had a very strong algebraic focus.

More recently, as the cadre of college-bound students has increased dramatically, the students can no longer be viewed as an elite group. Certainly, a comparable percentage of today's students are as good as the elite of the past, but these students likely attend the elite colleges. And, both today's elite students and the next tier of students have increasingly taken more sophisticated mathematics courses in high school, as we will discuss later.

When college algebra and precalculus courses were originally created, they were designed with the goal of preparing many of the, at that time, weaker students to go on to calculus in the sense of developing those algebraic skills that were necessary for success in calculus. At most schools today, these courses are still offered in the same spirit of being on the road toward calculus. But how well does this philosophy match reality?

First, let's consider the issue of why students take college algebra and related courses. In a study conducted at ten public and private universities in Illinois, Dunbar and Herriott [4] found that, typically, only about 10-15% of the students enrolled in college algebra courses had any intention of majoring in a mathematically intensive field. Similarly, Agras [1] found that only about 15% of the students taking college algebra at a very large two year college planned to major in mathematically intensive fields. And, we all know how quickly (and in which direction) courses such as college algebra can dramatically change the course of students' career intentions! So, the reality appears to be that only a small minority of students in these courses have a goal in which they would ever use the course content in the ways that we intend.

So, why do so many students take these courses? In general, these are the courses that are typically mandated to fulfill general education requirements or are required by other departments, most often by disciplines other than the traditional math intensive fields such as physics, engineering, and chemistry. Very few of the students take these courses because they enjoy mathematics or that their lives will be better in some way for having been exposed to more mathematics.

Moreover, data is beginning to emerge that provides a more detailed picture of just what actually happens to the students as a result of these courses. Dunbar [3] has tracked all students at the University of Nebraska-Lincoln for more than 14 years and has examined enrollment patterns among over 130,000 students. He found that only about 10% of the students who pass college algebra ever go on to start Calculus I and virtually none ever go on to start Calculus III. He has also found that about 30% of the students from college algebra eventually start business calculus. Weller [14] has confirmed these results at the University of Houston-Downtown, where 3.8% of the students who start college algebra ever go on to start Calculus I at any time over the following four years. McGowen [10] has found very comparable results at William Harper Rainey College, a large suburban two-year school outside Chicago. Consequently, it is clear that these courses, as presently constituted, do not meet the academic needs of the overwhelming majority of the students who take them.

It is clear that college algebra courses, as presently constituted, do not meet the academic needs of the overwhelming majority of the students who take them.

The fact is that college algebra and related courses are effectively the terminal course for the overwhelming majority of the students enrolled. Furthermore, the fact that virtually no students who take college algebra ever go as far as Calculus III means that virtually none of these students will be math majors, engineering majors, or majors in any other heavily quantitative field that requires more than a year of calculus.

Moreover, we are all well aware of the very low success rates in these courses, typically on the order of 50% and often considerably lower. Recently, in the provost's annual report at one of the largest two-year colleges in the country, the college algebra course was singled out as the one course that is most responsible for the school's losing students. Does anyone find this fact surprising? Similarly, the mayor in San Antonio likewise identified college algebra courses as the principal impediment to most college students' achieving a sufficiently high level of quantitative skills to function in the increasingly technological workplace that the mayor expects to develop in the city. As a result, he appointed a special task force consisting of representatives of all the local college math departments, as well as people from business and industry, to change college algebra to make it work. Now *that* is surprising!

Changes in the mathematical needs of students

Well, even if these courses do not serve the presumed need of preparing large numbers of students for calculus, perhaps they serve the needs of the other disciplines.

CRAFTY recently conducted a three year project in which leading educators from 17 quantitative disciplines met in a series of curriculum workshops to discuss and report to the mathematics community on today's mathematical needs of each discipline. The results of this *Curriculum Foundations* project, including the reports generated by each discipline workshop and overall recommendations generated in a summary workshop appear in [5].

In the past, the first mathematics course that appeared on the "radar screens" of the traditional, and the most math-intensive, quantitative disciplines (physics, chemistry, and engineering) was calculus. The introductory courses they offered were all calculus-based and so any course below calculus did not directly serve any of their needs. At most schools, these departments, especially physics and chemistry, now offer non-calculus-based versions of their introductory courses to much larger audiences than those who take the calculus-based courses. As a result, what students bring from precalculus and college algebra courses—and what they don't bring—is now a growing concern to the faculty in these other disciplines. The other quantitative disciplines represented in the Curriculum Foundations project, fields such as the life sciences, business and economics, and technology, typically require less mathematics of their students, so that courses at the college algebra level are the primary mathematical interest of the faculty in these areas.

There was an amazing degree of convergence of philosophy regarding these courses from all the disciplines. Perhaps most impressive is the fact that the identical recommendations came from almost all of the quantitative disciplines represented in the workshops. For instance, the main points made by the physicists were:

- *"Conceptual understanding of basic mathematical principles is very important for success in introductory physics. It is more important than esoteric computational skill. However, basic computational skill is crucial."*

- *"Development of problem solving skills is a critical aspect of a mathematics education."*

- *"Courses should cover fewer topics and place increased emphasis on increasing the confidence and competence that students have with the most fundamental topics."*

- *"The learning of physics depends less directly than one might think on previous learning in mathematics. We just want students who can think. The ability to actively think is the most important thing students need to get from mathematics education."*

- *"Students need conceptual understanding first, and some comfort in using basic skills; then a deeper approach and more sophisticated skills become meaningful. Computational skill without theoretical understanding is shallow."*

The engineers emphasized:

- *"One basic function of undergraduate electrical engineering education is to provide students with the conceptual skills to formulate, develop, solve, evaluate and validate physical systems. Mathematics is indispensable in this regard. The mathematics required to enable students to achieve these skills should emphasize concepts and problem solving skills more than emphasizing the repetitive mechanics of solving routine problems. Students must learn the basic mechanics of mathematics, but care must be taken that these mechanics do not become the focus of any mathematics course. We wish our students to understand various problem-solving techniques and to know appropriate techniques to apply given a wide assortment of problems."*

The business faculty recommended that:

- *"Mathematics is an integral component of the business school curriculum. Mathematics Departments can help by stressing conceptual understanding of quantitative reasoning and enhancing critical thinking skills. Business students must be able not only to apply appropriate abstract models to specific problems but also to become familiar and comfortable with the language of and the application of mathematical reasoning. Business students need to understand that many quantitative problems are more likely to deal with ambiguities than with certainty. In the spirit that less is more, coverage is less critical than comprehension and application."*

- *"Courses should stress problem solving, with the incumbent recognition of ambiguities."*

- *"Courses should stress conceptual understanding (motivating the math with the 'whys' —not just the 'hows')."*

- *"Courses should stress critical thinking."*

In a totally separate effort, a group of mathematicians from elementary school up through the university level is involved in a project to make quantitative literacy a major factor in everyone's education—in all disciplines and at all levels from elementary school up through college. As part of this initiative, leaders from business, industry, and government were brought together in the *Forum on Quantitative Literacy*. They discussed different perspectives on the issues of the mathematical preparation of students both for today's increasingly quantitative workplace and for the life-long ability for effective citizenship in today's society. Sentiments that were amazingly similar to those expressed through the Curriculum Foundations project were also voiced by these representatives from business, industry, and government. These views are enunciated very forcefully in articles in the volume, *Quantitative Literacy: Why Numeracy Matters for Schools and Colleges*, edited by Lynn Steen and Bernard Madison [13]. Moreover, of all the mathematics courses at the undergraduate level, the one that is most appropriate for developing the kind of quantitative literacy espoused by these people is college algebra—it has the greatest enrollment and potentially affects the students who most need that kind of experience.

But, if students do not need all the algebraic skills of the past, what do they really need in the way of mathematics today, let alone for tomorrow? Fifty years ago, virtually every mathematics problem in practice was continuous and deterministic. Problems with a discrete or stochastic (random) component were almost nonexistent. Basically, algebraic methods and differential equations with closed form solutions ruled! Today, the tables have turned 180^0—virtually every problem that arises is inherently discrete (in large part because of the digital age in which we live) and virtually every problem has some probabilistic component (there is always some uncertainty). But the mathematics curriculum, especially its first few years, has not changed appreciably to reflect the needs of the people who use mathematics today. This is particularly true at the large universities, where courses such as college algebra are often the only low-level, credit-bearing offerings; at many two-year colleges and some four-year schools, courses such as introductory statistics are given by the mathematics department as alternative offerings for some students not going into the sciences. However, even so, there are still many two-year schools and colleges where college algebra requirements force large numbers of students into college algebra and related courses who do not need much in the way of manipulative algebra for their majors.

The question we need to consider is: What should be the focus of mathematics education, especially at the level below calculus? Very few people today, let alone in the future, will need to factor anything as complicated as $x^8 - y^8$. However, virtually any educated individual will need the ability to

1. examine a set of data and recognize a behavioral pattern in it,
2. assess how well a given functional model matches the data,
3. recognize the limitations (often due to uncertainty) in the model,
4. use the model to draw appropriate conclusions, and
5. answer appropriate questions about the phenomenon being studied.

In turn, this process requires

• A deep understanding of the function concept, function notation, and the meaning of variable.

• A knowledge of different families of functions, including the ability to distinguish between the different families graphically, numerically and algebraically.

• A knowledge of the behavior of the different families of functions depending on one or more parameters.

• The ability to select the appropriate tool, be it pencil-and-paper, graphing calculators, spreadsheets, or CAS system, to solve the equations that arise from using the models.

• The ability to interpret the mathematical results and to communicate these ideas to others.

Let's see what these ideas mean in the context of college algebra courses and preparing students, not so much for calculus, but for courses in all other quantitative disciplines as well as for the ability to function as an educated citizen in an increasingly quantitative society. Perhaps the best way to see this is from the perspective of some of our critics—the faculty in other disciplines who express recurring complaints about what mathematical skills and knowledge students bring with them to those other courses and the students themselves who vote with their feet to abandon mathematics in such relentless numbers.

What then do students need to succeed in courses in other fields? They certainly need to know what a variable is, so that they can understand and use the formulas that arise. They need to know several fundamental classes of functions, most notably linear, exponential, logarithmic, and power functions. (Other than projectile motion, there are relatively few problems that lead to polynomials. Other than inverse proportions and inverse square laws, there are virtually no problems that lead to rational functions.)

Certainly, these are topics in standard college algebra courses, but they tend to get buried in a much more extensive array of techniques for factoring polynomials and producing graphs of every possible type of rational function. Is this really necessary? Not for the other disciplines. What about for calculus? Let's see where our years-long development of rational expressions and rational functions ends up. In order to find closed-form solutions for a handful of differential equations, such as the logistic equation $y' = ay - by^2$, one usually applies the method of partial fractions. (Ironically, there are simple ways to avoid the use of partial fractions altogether using a clever substitution to transform the differential equation into a simpler one that can be integrated easily without partial fractions; one can also utilize a CAS, if desired.)

But, to prepare for this, there is a heavy emphasis in traditional Calculus II courses on integration using partial fractions–often all four exhaustive (and exhausting) cases. To prepare for this, Calculus I courses often devote an inordinate amount of time to differentiating rational functions. To prepare for that, precalculus courses emphasize the behavior of all manner of rational functions and their graphs and occasionally even partial fraction decompositions. To prepare for that, college algebra courses emphasize the algebraic operations of adding, subtracting, multiplying, dividing, and reducing complex fractional expressions. Each of these is a hard algebraic technique that "separates the men from the boys". Is it any wonder that we see a 50% drop-off in mathematics enrollment with each subsequent course?

Is this what we want to do? If the techniques were so vital for success in subsequent courses (as was certainly true in the past) and if it was not possible to introduce what is needed on a timely basis in those subsequent courses to the small fraction of students who really need it, then a case can be made to include those topics. But these skills are no longer that important. Modern differential equations courses typically depend on computer software, including computer algebra system (CAS) to generate closed form solutions, so that the qualitative behavior of solutions and their dependence on initial conditions has become a far more important aspect of those courses. Similarly, there is now a strong emphasis on mathematical modeling to demonstrate the power of differential equations to provide understanding of a wide variety of natural processes.

The reality is that our students will rarely, if ever, have to integrate those terrible differential equations by hand. For the very few who will need to do it subsequently, would it be unreasonable to devote a few minutes to introducing the technique at the time? As a matter of fact, when solving differential equations, every experienced instructor already knows that he or she has to spend some time reviewing the techniques, because we know that the students have forgotten it altogether. So, do all of our students in the lower-level courses really need all that algebraic preparation? And, if they don't need all of it, what else can and should we do with the resulting available time in all our courses from developmental algebra up through calculus?

In the final analysis, few, if any, math departments can exist based solely on offerings for math and related majors and certainly none can exist at the two year college level. Whether we like it or not, mathematics is a service department at almost all institutions. And college algebra and related courses exist almost exclusively to serve the needs of other disciplines. If we fail to offer courses that meet the needs of the students in the other disciplines, those departments will increasingly drop the requirements for math courses. This is already starting to happen in engineering. Math departments, if they do not heed the voices of the partner disciplines and even become proactive to anticipate the changing needs of those fields, may well end up offering little beyond developmental algebra courses that serve little purpose.

Changes in School Mathematics

In 2004, about 225,000 students took one of the two AP calculus exams; reportedly, about twice as many took AP calculus in high school, but did not take the AP exam. In addition, many other students took non-AP calculus (often "polynomial calculus") in high school and a growing number are taking calculus either in an International Baccalaureate program or in a dual enrollment program to get credit from a local college. Together, this means that there are more students taking calculus in high school than in college.

Moreover, over the last decade, enrollment in college calculus courses has been, at best, stable and likely has decreased slightly. However, the number of students taking AP calculus in high school has been growing at an annual rate of about 8% over this past decade. Consequently, today's elite students are rarely seen in first year college calculus, let alone in precalculus or courses further down the collegiate mathematics sequence. Furthermore, the trends in calculus enrollment suggest that we can look forward to the day in the not-too-distant future when college calculus will become more of a remedial offering than the cornerstone of the undergraduate mathematics curriculum! Reportedly, several elite universities have already stopped giving college credit for first semester calculus.

Moreover, let's look at mathematics enrollments from a somewhat broader perspective. According to the 2000 Statistical Abstracts of the United States, 1,164,792 bachelor's degrees were awarded in 1996. Of these, only 13,143, or slightly over 1%, were in the mathematical sciences (which includes a large number in mathematics education). In the same year, 758 associate's degrees were awarded in mathematics out of a national total of 555,216 associate's degrees, which is on the order of one tenth of one percent. While we in the mathematics profession have a preoccupation with calculus in particular and the mathematics major in general, these offerings are only small potatoes at most colleges and universities. And by focusing on having the lower-level courses serve the needs of math majors, we tend to do a huge disservice to the overwhelming majority of the students we face.

Finally, an examination of the data in the CBMS surveys [8] and other studies of mathematics enrollments in both high school and college show a dramatic drop-off from one year to the next and one course to the next. Historically, about 50% of the mathematics audience is lost each year in high school. The efforts of NCTM over the last decade to keep students enrolled in mathematics longer has improved these figures dramatically from Algebra I to Algebra II; the drop-off rate is now only about 15%. But NCTM's efforts are in the same spirit as those recommended by AMATYC, MAA, and the partner disciplines. High school courses (including those in New York State) now feature a much greater emphasis on conceptual understanding and realistic problem solving, a lessened emphasis on algebraic manipulation for its own sake, and the routine use of graphing calculators in all aspects of the teaching and learning of mathematics.

In the process, there is a growing disconnect between the mathematics that students learn in high school (or below) and what mathematics is taught in the colleges, how that mathematics is taught, and the placement schemes that are used when students first arrive in college. There is something very disturbing about the juxtaposition of two trends: more students are taking more mathematics in high school, while simultaneously more students are being placed into developmental courses in college. The problem is that there is no longer a smooth transition from one level of education to the next and the students are suffering from it in huge numbers.

Here in New York, the problem is that few, if any, of the colleges ever adapted their mathematics offerings to reflect the Sequential 1-2-3 sequence or the new Mathematics A-B sequence. Worse, virtually all of the colleges in the state use either one of two national placement tests or home-grown placement tests that focus exclusively on determining how much in the way of traditional manipulative skills students possess. In the process, huge number so entering freshmen are placed in mathematics courses that are several semesters below what would be the comparable level they have achieved in high school. That is likely one of the major reasons for the growth in developmental mathematics at a time when students are required to take more mathematics in order to graduate from high school.

Technology and its implications for mathematics education

The student population that the majority of colleges face today consists predominantly of students who increasingly have not mastered traditional high school mathematics. In large part, this is because the emphasis in the high school classroom has been changing. The students are being taught more non-traditional mathematics—new topics (matrices, probability and statistics, etc), more emphasis on conceptual understanding, and more emphasis on mathematical modeling. However, all of the standard placement tests used in the colleges, as well as most home-grown tests, focus almost exclusively on identifying student weaknesses in algebraic manipulation. It is little wonder that ever greater proportions of students, who have experienced more and more mathematics in high school, are being placed in remedial tracks designed to develop all the traditional algebraic skills that once were necessary for a traditional calculus course.

But, freshman calculus courses have been undergoing significant change in the last decade as a result of the calculus renewal movement. These reform calculus courses seek to achieve:

- a balance among graphical, numerical, algebraic, and verbal approaches,
- an emphasis on conceptual understanding rather than rote manipulation, and
- a focus on realistic applications from the point of view of mathematical modeling, often through an early introduction to differential equations.

Much of this is possible because of the availability of sophisticated technology, most commonly graphing calculators, although some colleges make heavy use of computer software such as Derive, Maple or Mathematica with CAS (Computer Algebra System) capabilities.

Technology has not stood still since the advent of the graphing calculator. The first generation of graphing calculators, such as the TI-81 and the TI-85, essentially provided the tools to implement the graphical aspect of the Rule of Three – looking at functions from graphical, numerical, and symbolic perspectives. The second generation, such as the TI-82, the TI-83, and the TI-86 and similar models from other manufacturers provided additional tools to implement the numerical aspect through the use of lists, tables, and spreadsheet-like features. The newest generation, such as the TI-89, now complete the triad by providing the CAS capability to perform algebraic operations such as FACTOR, EXPAND, SIMPLIFY, SOLVE, DIFFERENTIATE, and INTEGRATE at the push of a button. They can solve, in a fraction of a second, any purely manipulative problem that we would ever have expected our very best students to do. When CAS capability was available only on a computer, it could perhaps be ignored as being too inconvenient to require of all students, except possibly in some advanced courses with limited enrollment. But, given the availability and reasonable prices of these new hand-held tools, we must face the challenge of rethinking the content, as well as the long-term value to the students, of any mathematics course that continues to place the development of traditional manipulative skills as its raison d'être.

The challenge we face is to find a reasonable balance between the use of technology and the level of algebraic skill development that is esential for utilizing the technology wisely.

But, if the students in college-level math courses are using sophisticated technology to assist in learning and doing mathematics, the practitioners who actually apply mathematics in all quantitative fields are utilizing technology that is at least as powerful as what we have available in the classroom. And this trend will undoubtedly expand as the capabilities of technology grow and the array of problems encountered outside the classroom expands in their level of sophistication and complexity.

In reality, any routine operation that people use repeatedly has already been programmed. It therefore makes little sense to offer mathematics courses that focus primarily on making students into imperfect organic clones of a $150 graphing calculator with CAS capabilities! The students will never win the competition—they will never be as fast or as accurate as the machine. Instead, we should be focusing on the intellectual and applied aspects of the mathematics that the machines cannot do.

The challenge we face is to find a reasonable balance between the use of technology and the level of algebraic skill development that is essential for utilizing the technology wisely.

What do our students really need?

We have mentioned a number of topics and themes that students need today for other disciplines and the workplace. Let's look at them in more detail.

1. The need for conceptual understanding

One of the most common complaints from faculty in other disciplines is that students do not know how to find the equation of a line. That is something we certainly teach, repeatedly, in every course in the curriculum. Just open any standard textbook from elementary algebra to precalculus and there are hundreds of problems that read: Find the equation of the line through the points (1,4) and (5,12). What more could the physicists, chemists, biologists, economists, etc. want?

Well, the problems that arise in their courses tend not to have just one digit, positive integers, for a start. The slope typically does not work out to be a one-digit integer or a simple fraction such as 1 at the worst. And, much more importantly, the faculty in the other disciplines tend not to give the students two simple points and tell them to create the equation. They expect their students to make a connection between the mathematics and the context, so that the equation and its component terms provide insight into the situation. They also expect their students to use the equation to answer questions about the context. Shouldn't their students be able to do that based on what we teach them in traditional courses?

In a recent article by F. S. Gordon [6], the answer to that last question turns out to be a resounding No! As one part of an extensive study comparing student performance, success rates, and attitudes based on the type of precalculus course—reform with a modeling emphasis or traditional with an algebraic emphasis—the department posed a series of common questions of a purely algebraic nature on final exams for both precalculus groups. One of these common questions had a contextual flavor. The students were given values for the enrollment at a college in two different years and were asked to find the equation of the linear function through those points and to give an interpretation of the meaning of the slope of the line in the context. In both groups, virtually every student could calculate the slope and find the equation of the line. In the reform group, virtually every student could give a meaningful interpretation to the slope. But in the traditional group, only about one-third could give a meaningful interpretation! A third left that part of the question out altogether; a large number simply restated the formula for the slope in words–the change in y over the change in *x*–but did not interpret the value or the context.

As the author put it, *"unless explicit attention is devoted to emphasizing the conceptual understanding of what the slope means, the majority of students are not able to create viable interpretations on their own. And, without that understanding, they are likely not able to apply the mathematics to realistic situations."* She goes on to address the broader implications of this finding. *"If students are unable to make their own connections with a concept as simple as that of the slope of a line (which they have undoubtedly encountered in previous mathematics courses), it is unlikely that they will be able to create meaningful interpretations and connections on their own for more sophisticated mathematical concepts."* We, and faculty in other disciplines, expect students to understand the significance of the base (growth or decay factor) in an exponential function. We expect them to comprehend what the parameters in a sinusoidal function tell about the phenomenon being modeled. We expect them to understand the significance of the derivative of a function and the significance of a definite integral. But, if students cannot create the connection between the slope of a line and its meaning in a context, it is clear that we should not expect them to create comparable connections of more sophisticated ideas on their own. It is our job to help them make those connections by emphasizing the meaning of the concepts, not just the formulas to be memorized and applied by rote.

One of the main themes of the calculus reform movement is an increased emphasis on conceptual understanding of the fundamental mathematical ideas and methods, not just a focus on the development of manipulative skills. (Recall the old adage: *You take calculus to learn algebra*.) This same principle of stressing conceptual understanding must be applied in the courses below calculus as well. If nothing else, we want the students to be prepared for calculus intellectually, not just algebraically. If they have not developed the ability to understand mathematical concepts and to value the importance of that understanding before walking into a calculus class, they are not prepared for a modern course in calculus. Nor are they prepared for any associated quantitative course in any other discipline that uses mathematical ideas.

In order to accomplish this, it is necessary to emphasize the importance of the concepts and this requires putting heavy emphasis on conceptual examples and problems, as opposed to primarily computational problems. And this emphasis must be put in classroom examples, in all homework assignments and on all exams. Homework problems should not be only repetitions of worked examples in the text that serve as templates. Exam problems should not be only further repetitions of what the students have previously seen. When a final exam is just a compilation of problems from class tests with the numbers changed, and when the class tests are just a compilation of weekly quizzes, students are not being educated. They are being trained in the same way that a seal at an amusement park such as Sea World is trained to answer such mathematical questions as: *How much is 2 plus 3?* The seal answers by slapping his flipper on the pool's deck until the trainer gives a hidden cue to stop.

For instance, many students come into college algebra or precalculus (or probably even into calculus) believing that you calculate the slope of a line by counting boxes—so many vertical boxes over so many horizontal boxes. This gimmick always worked in previous courses. But just change the scale, so that the vertical axis is measured in hundreds or thousands or millions and the horizontal scale is measured in 5-year increments and the box approach gives totally meaningless answers (unless you *understand* something about the size of the box). But if students have been trained, like the seal, to use this gimmick without the underlying understanding and they know that this gimmick gives the right answer, one has to emphasize strongly the basic ideas. And simply stating that the gimmick does not always work is not adequate; it must be accompanied by examples and problems with realistic contexts to drive home the point that more is needed than a simplistic trick.

We owe it to our students to do much more for them—not just for calculus or for other courses, but to function effectively in a rapidly changing society where the one thing they can and should expect is more change over the course of their careers. Simply put, no one will pay our students $30,000 or more a year if all they can do is reproduce solutions to problems memorized in high school and college mathematics classes!

2. The need for realistic problems

Reform calculus courses also usually include more realistic, and hence more sophisticated, problems and applications than routine problems that tend to be highly artificial. This theme should also be carried over to college algebra and precalculus courses. Traditional algebra applications such as "Ann is 8 years older than Billy and in 5 years she will be twice what he was 4 years ago" are not in the least realistic. There is no way to pose such a problem without knowing the ages in advance, which makes the entire problem totally artificial.

But what then constitutes a realistic problem? Just as two points determine a line, two points also determine an exponential function or a power function; three points determine a quadratic function; and so forth. In any realistic context, one can find two data points—just open a newspaper, a magazine, a textbook in any other quantitative field, or a copy of the Statistical Abstracts of the U.S. or search the Web. Presuming that the process being studied follows a linear or an exponential or a power function pattern, ask the students to find the equation of that function and use the resulting equation to answer some predictive questions in context.

All the algebra that anyone could want, and then some, is imbedded in answering those questions. Because the functions are based on real values, not artificially concocted one-digit integer values, the parameters are almost certain to be unpleasant decimal values, so there is plenty of opportunity to practice one's skills. But that practice is done in a hopefully interesting context. We are not asking the students to solve equations for the sake of practice, but to answer questions that they can see make sense to ask. That makes an incredible difference in terms of convincing the students that they are learning something that is potentially valuable to them.

For instance, consider the following set of data on the U.S. population, in millions.

Year	1950	1960	1970	1980	1990	2000
Population	150.7	179.3	203.3	226.5	248.7	281.4

Using technology, we can create the exponential function $P(t) = 85.38(1.0121)^t$ to model the growth of the population, where t is the number of years since 1900. We could then ask which is the independent variable and which is the dependent variable. (This may sound simplistic, but it is distressing how many students in precalculus find this challenging; in their experience, the independent variable is always x and there is nothing more to think about. But, without that understanding, there is no way that they can transfer the mathematics to other situations.) In addition, we can also give problems such as: Predict the U.S. population in 2005 using this model or When will the U.S. population reach 350 million? The latter requires solving the equation $85.38(1.0121)^t = 350$, a rather more daunting request than solving something like $5(2^x) = 80$, but it is far more interesting and useful. It is also far more akin to the kind of problems that students will face in their courses in other disciplines. We can also ask the students about the significance of the base, 1.0121, in the sense of the growth rate or we can ask them to suggest reasonable values for the domain and range of this function. Those concepts take on a very different life in a realistic context—it is no longer simply a matter of avoiding division by zero or points where one would take the square root of a negative number. And, more significantly, the development of reasoning skills and sound judgment are far more important characteristics for our students than the ability to factor a cubic polynomial.

Similarly, the following gives the total college enrollment, in millions, in various years.

Year	1955	1965	1965	1975
Enrollment (millions)	2.66	5.92	8.58	11.19
Year	1980	1985	1990	1995
Enrollment (millions)	12.10	12.25	13.82	14.95

We can create the power function $D(t) = 0.7345\, t^{0.8053}$ to model the college enrollment over time, where $t = 0$ in 1950. We could then ask questions such as: Predict college enrollment in 2010 or: When will there be 20 million people enrolled in college? The latter requires solving the equation $0.7345\, t^{0.8053} = 20$, which again is considerably more complicated, as well as considerably more meaningful, than the traditional type of problem such as solving $3x^4 = 48$.

The trigonometric functions can likewise be introduced using realistic situations. They serve as our primary mathematical models for periodic phenomena. For example, students can be asked to construct a sinusoidal function to model the temperature in a house where the furnace comes on when the temperature drops to 66? and turns off when the temperature reaches 70?, a cycle that repeats every 20 minutes. One possible result is $T = 68 + 2 \sin (\pi/10\ t)$, assuming that there is no phase shift. As another example, students can be asked to model a person's blood pressure over time given readings of 120 over 80 and a pulse rate of 70.

Each of these situations provides a wonderful opportunity to ask questions in context that go well beyond asking students to graph $y = 3 \sin 4x$ or to solve $6 \sin 2x = 3$ in terms of both the level of interest and the level of algebraic manipulation involved.

Furthermore, every such realistic problem carries with it the opportunity to reinforce the fundamental mathematical concepts—the meaning of the slope of a line or the growth or decay rate of an exponential function or the vertical shift, amplitude, period, and frequency of a sinusoidal function, etc. It also gives an opportunity to discuss domain and range issues repeatedly—how far can you reasonably extrapolate from the data points? What are the limitations of the model?

Alternatively, when we give a page-full of exercises asking the students to solve a collection of 50 or 100 equations that all look the same with the numbers changed, we send a very different message. In reality, only a handful of the students ever bother to do more than a small number of these problems.

To put this into perspective, I had a student in a modeling-based precalculus course several years ago who had a job as an airline mechanic at one of the major carriers. Before the course was finished, he was applying some of the ideas that were developed on identifying trends in data and making predictions based on the functions that result to study data at the airline on trends in the reduction of the level of maintenance on the planes. His work was then used by the mechanics union in the subsequent labor negotiations with management for a new contract. We *can* give our students knowledge and skills that they can apply immediately and directly if we change the focus in our courses. On the other hand, it is hard to envision any student coming out of a traditional college algebra or precalculus course who would be able to apply any of the ideas or techniques taught in such an immediate and significant way.

It is hard to envision any student coming out of a traditional college algebra or precalculus course who would be able to apply any of the ideas or techniques.

3. Other topics that should be emphasized

Another very common complaint from the other disciplines is that students do not have any understanding of or facility with exponents and logarithms. In partial response to this, the calculus reform projects have placed considerably more emphasis on exponential and logarithmic functions. They are no longer relegated to a chapter at the beginning of Calculus II, but have been brought up front as some of the fundamental functions of mathematics. The same kind of emphasis is required in the courses below calculus, not just to prepare the students for the subsequent calculus experience, but perhaps even more importantly because these functions are so vital in all quantitative disciplines today.

In that regard, the treatment of these functions in college algebra and precalculus should not be just one stand-alone chapter and the functions never reappear. Instead, exponential functions and their properties, just like linear functions and their properties, should arise repeatedly in many different contexts throughout the course. If we want students to develop an appreciation for certain ideas, we have to give more emphasis to those concepts; if every topic or type of function receives equal attention, students do not learn what is important and, at best, make their own decisions of what they should learn for the long haul.

Furthermore, in the other disciplines, the various mathematical functions typically arise in the context of finding an appropriate function to model sets of data, just as they were applied in the above illustrations. There is a reason that these curve-fitting techniques are incorporated into all graphing calculators and spreadsheets such as Excel—they are the standard tools of today's practitioners, both in class and on the job. But, it is not as simple a matter as just pushing a button to get an answer. Some very deep levels of understanding are essential. One has to know the behavioral characteristics of each family of function in order to make an intelligent selection of possible functions to use as models. There are some critical difficulties that can arise that are domain issues for these functions. For instance, the routines used by calculators and by spreadsheets to fit exponential, power, and logarithmic functions to data involve transformations of the data to plot log y versus x, or log x versus y, or log y versus log x. (The first two are semi-log plots; the third is a log-log plot.) But, if any of the data entries is zero or negative, the logarithms are not defined. The error messages the systems give are not exactly self-explanatory; the person who pushes the button has to know the mathematics to understand the message and to *know* how to avoid the problems.

These techniques and ideas are ideal ones to incorporate into college algebra and precalculus classes for a variety of reasons. First, they give the opportunity to reinforce the important characteristics about each family of function, so that the students see the ideas coming back again. Second, they see how these functions arise in practical settings, which is a great motivation for topics that otherwise tend not to appear all that useful. Third, this gives us the opportunity to ask interesting, predictive questions in the contexts of the data, so that the students have even more occasions to practice their skills solving the resulting equations. Fourth, the students are being prepared for the specific kinds of applications that will arise in their other courses; in turn, this increases their level of appreciation for the mathematics course. That may not sound terribly important, but in the long run, it makes our courses far more important to the students. Instead of dropping out of mathematics, they are encouraged to continue to subsequent courses.

The mathematical topics discussed above are already standard topics in college algebra and precalculus courses. It is the emphasis on real-world data and applications that make the difference. However, there are other topics that typically are not treated in these courses at all which are of importance to the other disciplines. For instance, most disciplines today want their students to know something about statistics. They usually need their students to have some concept of probability, so that they can use random simulations to model complex processes. Many disciplines would like their students to know about recursion. And, a growing number of disciplines would like their students to be able to read and interpret contour plots, something that we in mathematics would not dream of introducing in any course below Calculus III. All of these things are challenges we have to find ways to face if we are to keep our clients happy and provide the mathematical experiences large numbers of our students need.

What can be removed to make room?

Clearly, there are many new topics and methods that can and should be included in courses at this level. To do so, we have to find time. This means eliminating something (or lots of somethings).

Back when the author studied trigonometry, we were all expected to know three fundamental laws: the Law of Sines, the Law of Cosines, and the Law of Tangents. These were not just theorems or formulas; they were *Universal Laws*! Today, however, virtually no mathematician knows about the Law of Tangents, let alone can quote it. It is not that this law was repealed or that triangles stopped obeying it. The reality is that many topics, some of marginal significance, others that once were considered extremely important (otherwise the **Law** of Tangents would not have been called a law), have been removed from the curriculum in the past with seemingly minimal long-term impact.

We face the same decisions today. Some topics in the present syllabus have to be relinquished to make room for newer, more important topics. Over the last half century, as has been pointed out, the focus of mathematics in practice has changed dramatically and an incredible body of new mathematical ideas and techniques has been developed. In turn, we owe it to our students to at least acquaint them with some of these concepts—matrix algebra, statistical ideas, probabilistic reasoning via simulation, recursion and difference equations, etc.—early in their mathematical experiences. Part of the need is to provide the students with a broader view of what mathematics is all about; more importantly, these are important mathematical techniques they will need for their other courses.

For example, in the 1930s, linear algebra was a graduate course; it gradually worked its way down to a junior-senior offering, then to a sophomore-level course, and today matrix methods and their applications are standard topics in modern high school mathematics. Once, the entire focus of algebra up through calculus was to prepare students for a traditional course in differential equations where they saw, for the first time, the power of the mathematics to create mathematical models and to formulate closed-form solutions. Today, the focus in most other disciplines is on difference equations instead of differential equations—they are conceptually easier, they are simpler to set up, and they are much easier to solve, numerically and graphically, with modern technology.

What then can be relinquished from precalculus and college algebra courses? One possibility is to downplay the emphasis on rational expressions and rational functions at all levels of the curriculum, as suggested before. Several other topics that could be eliminated are things like Descartes' rule of signs, the rational root theorem, and synthetic division. For a long time, when finding the roots of a polynomial was a major undertaking, these were valuable tools of the mathematician and the practitioner. Today, many students have calculators with a FACTOR button. Every student has (or should have) at his or her fingertips a calculator with a numerical root-finding routine built in, not to mention the ability to zoom in repeatedly either on the graph or in a table of values associated with any polynomial. Locating the real roots of a polynomial is no longer a challenge; it should not be a major emphasis in our courses. But understanding what the roots are and knowing how to use them intelligently is certainly still a critically important aspect of these courses.

Another case in point has to do with solving systems of linear equations. Many instructors still have their students solve 2 x 2 and 3 x 3 systems using Cramer's rule. Most have their students solve such systems using various algebraic techniques, such as the method of substitution or the method of elimination. Students should see these methods (they almost certainly have seen them in high school, for that matter), at least in simple cases. But, is it really worth the time of having them practice doing this over and over with dozens of examples and problems when they possess technology that will solve a system of 99 equations in 99 unknowns in far less time than they can solve a system of two equations in two unknowns by hand? We need to find the right balance between hand-experience and use of technology here.

In a similar vein, we have six trigonometric functions because, in the pre-technology centuries, it was much simpler to have tables of values available for all six possible ratios of the sides in right triangles to minimize hand computation. Today, such computational issues are irrelevant. So, what valuable roles do the secant, the cosecant, and the cotangent play? There are very few realistic problems that involve any of these functions and each of these problems can be solved quite easily by using only the sine, the cosine, and the tangent, along with several "new" identities:

$$1 + \tan^2 x = \frac{1}{\cos^2 x}$$

and

$$\frac{d(\tan x)}{dx} = \frac{1}{\cos^2 x}.$$

The Harvard calculus course, for instance, totally avoids the use of cotangent, secant, and cosecant without any loss; there is therefore little reason to bedevil students with them in precalculus and trigonometry courses. They take up an inordinate amount of time for virtually no gain. In fact, a number of other nations, including Russia, France, and Israel, apparently never mention these three functions in any of their mathematics courses and the mathematicians and scientists they produce never seem particularly handicapped by this loss.

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